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GENERAL COMPETITIVE EQUILIBRIUM OF THE SPATIAL ECONOMY

Two Teasers*

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The local indivisibility of space complicates general equilibrium analysis. Mazzoleni and Montesano (1984, p. 286), suggest that the usual reasoning of the non-spatial analyses of competitive equilibrium can be performed in the context of a commodity space that contains subsets of land. This space, however, lacks a linear structure, so it resists application of the usual reasoning. Two examples will show that Mazzoleni and Montesano's existence results are flawed.

1. Introduction

This note examines general equilibrium problems when subsets of a geographical space are commodities. This modeling is highly appropriate for spatial economics and, in particular, takes into account the local indivisibility of space. The consequent commodity space lacks a linear structure as the acts of splitting or combining produce different commodities. This defect invalidates standard general equilibrium results such that they cannot be repaired along the simple lines suggested in a recent paper by Mazzoleni and Montesano (1984). Their framework is summarized in the next section. Sections 3 and 4 present examples that invalidate existence of demand and equilibrium in the spatial context of Mazzoleni and Montesano. Section 5 concludes.

2. The framework

Mazzoleni and Montesano (1984) present a rather elaborate general

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equilibrium model of the spatial economy. The punch of the present argument is easiest received when the model is stripped of non-space commodities, producers, and the distinction between use and property of space, to her bare bones of space commodities. Then a commodity is described by a subset, F_j, of (geographical) space A of the economy under examination, where index j ($j=1,\ldots,N$) indicates an agent. By representing with $\mathscr{J}$ the set which is composed of all the subsets of A, we have $F_j \in \mathscr{J}$ for any $F_j \subseteq A$. Thus, $\mathscr{J}$ is the commodity space.

Consumer j has a utility function U_j over land parcels. The existence of a *continuous* utility representation of a preference ordering over a subset of an infinite dimensional space is itself a difficult matter, since the underlying space might not have the appropriate topological properties; see Debreu (1964).

Consumers maximize utility, $U_j(F_j)$, subject to the budget constraint. Expenditure on land is $\int_F r(P)\,\mathrm{d}P$, r the rent density, and may not exceed the value of the initial endowment. Note that r must be summable and F measurable. This is obtained by restricting $\mathscr{J}$ to the natural Borel field on A. $\mathscr{J}$ is also assumed to have a 'suitable' topology [Mazzoleni and Montesano (1984, p. 294)]. U_j is supposed to be continuous and super-additive. Here continuity is with respect to the 'suitable' topology, while superadditivity is determined through the following definitions:

- a set $H \subseteq \mathscr{J}$ is said to be convex iff for any pair $F_1, F_2 \in H$ with $F_1 \subseteq F_2$ there exists an increasing function D: $[0,1] \to \mathscr{J}$ with $D(0)=\varnothing$, $D(1)=F_2-F_1$, such that $F_1 \cup D(t)$ and $F_2 - D(t) \in H$ for $0 \leqq t \leqq 1$.
- U is superadditive on convex H iff for any pair $F_1, F_2 \in H$ with $F_1 \subseteq F_2$ there exists an increasing function D: $[0,1] \to \mathscr{J}$ with $D(0)=\varnothing, D(1)=F_2-F_1$, such that

$$U[F_1 \cup D(t)] \geqq (1-t)U(F_1) + tU(F_2),$$

$$U[F_2 - D(t)] \geqq tU(F_1) + (1-t)U(F_2).$$

Utility maximization yields optimal solutions F_j^* depending on the rent density. Since solutions need not be unique, a demand correspondence is defined from rent densities to the commodity space, $\mathscr{J}$. In equilibrium, the rent density is such that F_j^* exist that partition all available land.

3. A demand teaser

The first theorems of Mazzoleni and Montesano (1984) bear on the demand correspondence and amount to an analysis that 'guarantees that the collections of the optimal solutions for (production and) exchange agents are non-empty, convex and upper semicontinuous'. This is false. The theorems do not prove the existence of an optimal solution and, in fact, a counter-example can be given.

Consider the following set of commodities:

$$H=\left\{F_n=\left[0,\frac{1}{n}\right]\cup\left[\frac{2}{n},\frac{3}{n}\right]\cup\left[\frac{4}{n},\frac{5}{n}\right]\cup\cdots\cup\left[\frac{n-2}{n},\frac{n-1}{n}\right]\middle| n=1,2,\ldots\right\}\subseteq\mathscr{I},$$

the commodity space of subsets of the unit interval. Note that no F_n is contained in any F_m, so H is trivially convex and any utility function U is trivially superadditive in the sense of section 2. So the only condition imposed by the Mazzoleni and Montesano framework is through the continuity of U. As far as H is concerned, it suffices to require that $U(F_n)$ has a limit. Now let this limit be approached from below; in other words, F_n is better for increasing n. Let the supremum of $U(F_n)$ also be the supremum utility level given the budget, for example when the budget can cover half the unit interval and the rent density is uniform. Then the limit set is the commodity demanded. But what is the limit of F_n? Candidates are the unit interval itself, half the indicator function thereon, and the empty set. They indeed arise when alternative topologies on $\mathscr{I}$ are imposed [Berliant and ten Raa (1984)]. Each candidate limit can be ruled out as optimal solution, however. The unit interval violates the budget as it is two times too big. Half the indicator function does not belong to the commodity space, $\mathscr{I}$. The empty set can be no solution when monotonicity is allowed, as, for example, in Theorem 4 of Mazzoleni and Montesano (1984). A limit that has none of these defects does not exist. Demand is undefined. Note that the perhaps most intuitive limit, half the indicator function, is ruled out because $\mathscr{I}$ lacks a linear structure since commodities are indivisible. Also implicit here is that compactness in infinite dimensional spaces is difficult to guarantee.

4. An equilibrium teaser

Ignore the problem of demand and turn to equilibrium. Mazzoleni and Montesano (1984) follow the usual reasoning to demonstrate the existence of equilibrium, by an extension of the Gale–Nikaido theorem. Basically, this theorem is proved by sending one price–commodity pair into other ones through the demand correspondence and the response function that prices excess supply as low as possible. A fixed point of this combined mapping is shown to be an equilibrium. In fact, the Gale–Nikaido theorem is equivalent to Kakutani's theorem [Hoàng (1976)]. The last theorem of Mazzoleni and Montesano extends this fixed point argument to correspondences on $\mathscr{I}$ instead of the usual Euclidean commodity space. The mathematical requirements of continuity and compactness are now assumed with respect to the 'suitable' topology, while the requirement of convexity is assumed in the sense of section 2. The continuity and compactness conditions are obscure and intractable, but it is known that they can be extended to infinite dimensional cases such that the Gale–Nikaido theorem holds [Florenzano (1984)]. The convexity condition is plainly insufficient, as a simple example will illustrate.

Take any geographical space, A, with interior; let, as before, $\mathscr{J}$ consist of subsets of A, and define $f: \mathscr{J} \to \mathscr{J}$ by $f(E) = \overline{A - E}$. Note that f is single-valued. The empty set may be assumed as a value, but this is a relevant member of $\mathscr{J}$, especially when the budget is zero or, more generally, when land is priced out of the market. The only reason that f is closed-valued is to make it compatible with compactness requirements. f fulfills all the fixed point assumptions of Theorem 4 of Mazzoleni and Montesano (1984). In particular, convex-valuedness in the sense of section 2 is trival as f is single-valued.

Yet f has no fixed point. Suppose, to the contrary, that $f(E) = \overline{A - E} = E$. Then $E \subseteq \overline{A - E}$, so that each E-member is approached by non-E-members, by which the interior of E is empty. On the other hand, $A - E \subseteq E$, so that $A = E \cup A - E \subseteq E \cup \overline{A - E} \subseteq E \cup E = E$. It follows that the interior of A is empty, a contradiction.

The fixed point argument fails in the present context of land, because the commodity space, $\mathscr{J}$, lacks a linear structure. As far as the structure plays a role in the nature of values of correspondences, such as convexity, one might be able to substitute a mere set ordering based property, such as convexity in the sense of section 2. But the absence of a linear structure is much more damaging to the topology of the domain of the correspondence, a fact that is completely ignored by Mazzoleni and Montesano.

A further illustration of the non-existence of equilibrium is as follows. The basic idea is that Mazzoleni and Montesano (1984) do not rule out demand consisting of a single point of A. Let the topology be favorable to the issue addressed by the first three of their theorems, regarding optimal solutions for production and exchange agents, say the one proposed by Berliant and ten Raa (1984). A simple utility function that observes the weak conditions of Mazzoleni and Montesano is defined by $U(E) = \inf E$, where E is situated in A, the unit interval. Then agents will demand a single point, common to all, no matter rent density. But at this point there will be excess demand, so that equilibrium does not exist. The existence of equilibrium, or, mathematically speaking, a fixed point, requires more than Theorem 4 of Mazzoleni and Montesano (1984) suggests.

One requirement of Theorem 4 is that the set of land prices (measures) be a compact subset of a normed linear space. Since the natural topology of a normed linear space is very strong, only a few subsets are compact and the likelihood that one of them contains equilibrium prices is small. Let us explain. Mazzoleni and Montesano's requirement that the price set be compact in the normed space implies, via Munkres (1975, Corollary 8.4), that it is locally compact Hausdorff. In essence, this means that the price set has finite dimension. More precisely, if the price set contains a neighborhood of the origin of the normed linear space, then it is a locally compact topological vector space and hence, via Rudin (1973, Theorem 1.22), of finite dimension. Consequently, prices must be expressible as linear combinations of a fixed, finite set of basis elements. The likelihood of finding equilibrium prices in the set supporting an equilibrium allocation of a model with infinitely many commodities is small.

5. Conclusion

The demand and equilibrium existence results for a spatial economy obtained by usual general equilibrium analysis reasoning [Mazzoleni and Montesano (1984)] are false. The local indivisibility of land complicates to an extent that cannot be remedied by simple redifinition of the mathematical conditions of continuity, compactness, and convexity. The very topology of the commodity space is so weakened that the fixed point property is invalidated. Proof by analogy does not work in the non-linear infinite dimensional context. Research should be directed towards the spotting of additional conditions on production and utility functions needed for the existence of spatial equilibrium.

Conditions sufficient to guarantee the existence of demand can be found in Berliant (1984) and Berliant and ten Raa (1984). It is well-known that the conditions sufficient to guarantee the existence of an equilibrium with infinitely many commodities [see Bewley (1972)] or indivisibilities [see Svensson (1983)] are restrictive. Berliant (1985) has made some progress in this direction.

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